Singular Value and Generalized Singular Value Decompositions and the Solution of Linear Matrix Equations

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ABSTRACT

The solution of the linear matrix equations (i) $AXB+CYD=E$ and (ii) $(AXB,FXG)=(E,H)$ are considered. New necessary and sufficient conditions for the consistency of the equations are derived, some using the generalized singular value decomposition. Special cases (iii) $AX+YD=E$ and (iv) $AXB=E$ are treated using the singular value decomposition. Numerical algorithms for the solutions are also suggested.

1. INTRODUCTION

Let $\mathbb{R}^{m\times n}$ denote the space of real $m\times n$ matrices.

We consider the solution of the linear matrix equations

$$AXB+CYD=E \tag{1}$$

with $A\in\mathbb{R}^{m\times p}$, $B\in\mathbb{R}^{q\times n}$, $C\in\mathbb{R}^{m\times r}$, and $D\in\mathbb{R}^{s\times n}$; and

$$AXB=E, \tag{2a}$$

$$FXG=H, \tag{2b}$$

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with $F \in \mathbb{R}^{\tilde{m} \times p}$ and $G \in \mathbb{R}^{q \times \tilde{n}}$. We also consider the special cases

$$AX + YD = E \tag{3}$$

and

$$AXB = E \tag{4}$$

of Equations (1) and (2) respectively.

For Equation (3), the necessary and sufficient conditions for its consistency have been derived in [1] and [9] (and some references therein). For inconsistent equations, l_p and Chebyshev solutions can be considered [11, 12]. For Equation (4), the consistency conditions were given by Penrose (see [7]). If Equation (3) or (4) is consistent, a solution can be obtained using generalized inverses (GI) [1, 7]. In Sections 4 and 5, the singular value decomposition (SVD) [4] will be used to investigate Equations (3) and (4), and a simple and clear exposition, in terms of consistency conditions, analytic and numerical solutions, and l_2-solutions, will be given.

For the general cases in (1) and (2), consistency conditions were given in [2] and [6] respectively, and solutions were again given in terms of GI. Simpler conditions for the consistency of Equations (1) and (2) can be obtained through the use of the generalized singular value decomposition (GSVD) ([4], [8], [10], and references therein), and numerical algorithms for their solutions arise naturally; the results are contained in Sections 3 and 6 respectively.

Many authors have not been aware of the fact that Equations (1) and (2) are dual in some sense. The duality is discussed in Section 7.

The paper is completed with a brief introduction to SVD and GSVD in Section 2, and a conclusion in Section 8.

Note that an excellent thesis on the general equation

$$\sum_{j=1}^{N} A_{ij} X_j B_{ij} = C_i, \qquad i = 1, \ldots, M,$$

can be found in [5]. Questions of consistency, near-consistency (an equation was defined to be ε-consistent iff the residual is less than ε), and applications to output feedback pole assignment problems in control theory were considered. Some results in Sections 3 and 6 were obtained (in less elegant forms) in [5] as the GSVD was used tacitly.

Other linear matrix equations have been considered by the author in [3], and some more results will appear in future papers.

Note that the existence results of Sections 3 and 4 are closely related to those in [1] and [2].

2. SVD AND GSVD [4, 8, 10]

Given a matrix $A \in \mathbb{R}^{m\times n}$ of rank k, one has

$$A = UDV^T, \tag{5}$$

where $U = (U_1, U_2)$ and $V = (V_1, V_2)$ are orthogonal matrices, with

$$D = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \tag{6}$$

and $\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_k)$, $\sigma_i > 0$.

It is easy to see that the matrices U_1, V_1, V_2, and U_2 span the range and null spaces of A and A^T respectively. Other properties of the SVD can be found in standard texts such as [4], and references therein.

The GSVD, a generalization of the SVD, can be described as follows [8]: Given two matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times n}$ with the same number of columns, there exists orthogonal matrices U and V and nonsingular X such that

$$A = U\Sigma_A X, \qquad B = V\Sigma_B X, \tag{7}$$

where $\Sigma_A \in \mathbb{R}^{m\times n}$, $\Sigma_B \in \mathbb{R}^{p\times n}$, and

$$k = \mathrm{rank}(C) = \mathrm{rank}\begin{pmatrix} A \\ B \end{pmatrix},$$

with

$$\Sigma_A = \begin{array}{c} \left(\begin{array}{ccc|c} I_A & & & \\ & S_A & & 0 \\ & & 0_A & \end{array}\right) \\ \begin{array}{cccc} r & s & k-r-s & n-k \end{array} \end{array}, \tag{8a}$$

$$\Sigma_B = \begin{array}{c} \left(\begin{array}{ccc|c} 0_B & & & \\ & S_B & & 0 \\ & & I_B & \end{array}\right) \\ \begin{array}{cccc} r & s & k-r-s & n-k \end{array} \end{array}; \tag{8b}$$

here I_A and I_B are identity matrices, 0_A and 0_B zero matrices, and

$$S_A = \mathrm{diag}(\alpha_1, \ldots, \alpha_s), \tag{9a}$$

$$S_B = \mathrm{diag}(\beta_1, \ldots, \beta_s), \tag{9b}$$

with $1 > \alpha_1 \geqslant \cdots \geqslant \alpha_s > 0$, $0 < \beta_1 \leqslant \cdots \leqslant \beta_s < 1$, and $\alpha_i^2 + \beta_i^2 = 1$, $i = 1, \ldots, s$.

Some submatrices in Equation (8) can vanish, depending on the structures of the matrices A and B.

Proofs and properties concerning the GSVD can be found in [4], [8], [10], and references therein.

In situations where the matrix X in Equation (7) has to be inverted (e.g. in Sections 3 and 6), ill-conditioning may occur, as the matrix X is not orthogonal. In [8], the matrix is expressed as

$$X = Q\begin{pmatrix} R^{-1}W & 0 \\ 0 & I \end{pmatrix} \tag{10}$$

where the matrices Q and W are orthogonal, and

$$C = \begin{pmatrix} A \\ B \end{pmatrix} = P\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}Q^T, \tag{11}$$

with the matrix P being orthogonal. [It can be the SVD in Equation (11).] Thus, the matrix X will be ill conditioned if the smallest nonzero singular value of C is small, i.e. when the (numerical) rank determination of the matrix C is not straightforward.

A stable numerical algorithm for the computation of the GSVD by Stewart can be found in [10].

3. $AXB + CYD = E$

Decomposing the matrix pairs (A^T, C^T) and (B, D) using the GSVD, Equation (1) is equivalent to

$$X_1^T \Sigma_A^T U_1^T \cdot X \cdot U_2 \Sigma_B X_2 + X_1^T \Sigma_C^T V_1^T \cdot Y \cdot V_2 \Sigma_D X_2 = E, \tag{12}$$

where the matrices U_i and V_i are orthogonal, and X_i are nonsingular, as in Equations (7) to (9).

Define $\tilde{X} = U_1^T X U_2$, $\tilde{Y} = V_1^T Y V_2$ and $\tilde{E} = X_1^{-T} E X_2^{-1}$. Equation (12) now reads

$$\Sigma_A^T \tilde{X} \Sigma_B + \Sigma_C^T \tilde{Y} \Sigma_D = \tilde{E}. \tag{13}$$

Note that transforming Equation (12) to (13) does not change the equation's consistency.

Partitioning the matrices $\tilde{X}, \tilde{Y}$ and $\tilde{E}$ according to the Σ's, Equation (13) is equivalent to

$$\begin{pmatrix} I_A & & \\ & S_A & \\ & & 0_A^T \\ \hline & 0 & \end{pmatrix} \tilde{X} \left(\begin{array}{ccc|c} I_B & & & \\ & S_B & & 0 \\ & & 0_B & \end{array}\right)$$

$$+ \begin{pmatrix} 0_C^T & & \\ & S_C & \\ & & I_C \\ \hline & 0 & \end{pmatrix} \tilde{Y} \left(\begin{array}{ccc|c} 0_D & & & \\ & S_D & & 0 \\ & & I_D & \end{array}\right) = \tilde{E},$$

[c.f. Equation (8)], which in turn is equivalent to

$$\begin{pmatrix} \tilde{X}_{11} & \tilde{X}_{12} S_B & 0 & 0 \\ S_A \tilde{X}_{21} & S_A \tilde{X}_{22} S_B + S_C \tilde{Y}_{22} S_D & S_C \tilde{Y}_{23} & 0 \\ 0 & \tilde{Y}_{32} S_D & \tilde{Y}_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \tilde{E}$$

$$= \left(\begin{array}{ccc|c} \tilde{E}_{11} & \tilde{E}_{12} & \tilde{E}_{13} & \\ \tilde{E}_{21} & \tilde{E}_{22} & \tilde{E}_{23} & \tilde{E}_{\cdot 4} \\ \tilde{E}_{31} & \tilde{E}_{32} & \tilde{E}_{33} & \\ \hline & \tilde{E}_{4\cdot} & & \tilde{E}_{44} \end{array}\right). \tag{14}$$

The following theorem is a consequence of Equation (14):

Theorem 1. *Equation* (1) *is consistent if and only if the following submatrices of* $\tilde{E}$ *vanish*:

$$\tilde{E}_{13}, \tilde{E}_{31}, \tilde{E}_{4\cdot}, \tilde{E}_{\cdot 4}, \tilde{E}_{44}.$$

For consistent equations, the submatrices

$$\tilde{X}_{13}, \tilde{X}_{23}, \tilde{X}_{33}, \tilde{X}_{32}, \tilde{X}_{31}; \qquad \tilde{Y}_{13}, \tilde{Y}_{12}, \tilde{Y}_{11}, \tilde{Y}_{21}, \tilde{Y}_{31}$$

of $\tilde{X}$ and $\tilde{Y}$ respectively are arbitrary, and additional degrees of freedom can be found in

$$\tilde{X}_{22} = (\tilde{x}_{ij}) \quad \text{and} \quad \tilde{Y}_{22} = (\tilde{y}_{ij}).$$

Elementwise, one has

$$\begin{pmatrix} \tilde{x}_{ij} \\ \tilde{y}_{ij} \end{pmatrix} = M_{ij}^{+}\tilde{e}_{ij} + \left(I - M_{ij}^{+}M_{ij}\right)Z_{ij}, \tag{15a}$$

where $S_A = \operatorname{diag}(\alpha_i)$, $S_B = \operatorname{diag}(\beta_i)$, $S_C = \operatorname{diag}(\gamma_i)$, $S_D = \operatorname{diag}(\delta_i)$, *and* $\tilde{E} = (\tilde{e}_{ij})$; $M_{ij} = (\alpha_i\beta_j, \gamma_i\delta_j)$; $(\cdot)^{+}$ *denotes the* (1,2,3,4,)*—or Penrose*—GI [7]; *and the vectors Z_{ij} are arbitrary.*

If we consider $\tilde{Y}_{22}$ to be arbitrary, then $\tilde{X}_{22}$ must be chosen to be

$$\tilde{X}_{22} = S_A^{-1}\left(\tilde{E}_{22} - S_C\tilde{Y}_{22}S_D\right)S_B^{-1}. \tag{15b}$$

(*If we consider $\tilde{X}_{22}$ to be arbitrary, we choose $\tilde{Y}_{22}$ similar to Equation* (15b) *accordingly.*)

Proof. The consistency conditions, the arbitrariness of the submatrices, and Equation (15b) are trivial from Equation (14).

For $\tilde{X}_{22}$ and $\tilde{Y}_{22}$, consider the (i, j) component $\tilde{x}_{ij}$ and $\tilde{y}_{ij}$; Equation (14) implies

$$M_{ij}\begin{pmatrix} \tilde{x}_{ij} \\ \tilde{y}_{ij} \end{pmatrix} = \tilde{e}_{ij}$$

and thus Equation (15a). Note that, from the definition of GSVD and Equation (9), the row vectors M_{ij} are nonzero and right-invertible. ∎

Using Theorem 1 and Equation (15b), one characterization of the solution of a consistent equation (1) will be

$$X = U_1\begin{pmatrix} \tilde{E}_{11} & \tilde{E}_{12}S_B^{-1} & Z_2 \\ S_A^{-1}\tilde{E}_{21} & S_A^{-1}\left(\tilde{E}_{22} - S_C Z_1 S_D\right)S_B^{-1} & Z_3 \\ Z_4 & Z_5 & Z_6 \end{pmatrix}U_2^T$$

and

$$Y = V_1 \begin{pmatrix} Z_7 & Z_8 & Z_9 \\ Z_{10} & Z_1 & S_C^{-1}\tilde{E}_{23} \\ Z_{11} & \tilde{E}_{32}S_D^{-1} & \tilde{E}_{33} \end{pmatrix} V_2^T,$$

where the matrices Z_1 to Z_{11} are arbitrary.

Equation (14) leads naturally to a numerical algorithm for the solution of a consistent equation (1). The process will then be numerically unstable [and Equation (1) numerically ill conditioned] if any of the generalized singular values (GSV) α_i, β_i, γ_i, and δ_i is small, or the matrix X_1 or X_2 ill conditioned. [Cf. discussion after Equation (11).] Small GSV may have to be reset to zero, with the resulting errors treated as residuals of Equation (1).

It is important to keep in mind that the numerical algorithm has not been proved to be numerically stable in the conventional sense, as a backward error analysis is not yet available. Furthermore, the transformation of E to $\tilde{E}$ in Equation (13) invariably loses information, because of the inversions of the nonorthogonal matrices X_1 and X_2. The problems can be avoided by using the direct Kronecker product approach [4], which involves a bigger matrix, but inherits the backward error analysis of the Gaussian elimination process. Thus, the possible instability related to the inversions of X_1 and X_2 in Equation (13) is a price we pay for the reduction of the size and complexity of the problem.

The transformations by orthogonal matrices of $\tilde{X}$ and $\tilde{Y}$ back to the solution of X and Y will be well conditioned [4].

By choosing various arbitrary matrices [ignoring Equation (15b)] to be zero, a solution of least 2- or F-norm may be obtained, but the residual of an inconsistent equation (1) will not be of least norm, as X_1 and X_2 in Equations (12) and (13) are not orthogonal.

4. $AX + YD = E$

For the special case (3) of Equation (1), it is not necessary to use the GSVD for the solution. [The GSVD of (A, I) degenerates into the SVD of A.]

Decomposing the matrices A and D by the SVD, Equation (3) is equivalent to

$$\begin{aligned} & U_A D_A V_A^T \cdot X + Y \cdot U_D D_D V_D^T = E \\ \Leftrightarrow \quad & D_A \tilde{X} + \tilde{Y} D_D = \tilde{E} \end{aligned} \tag{16}$$

with $\tilde{X} = V_A^T X$, $\tilde{Y} = YU_D$, and $\tilde{E} = U_A^T E V_D$. Note that all the transformations involved in equation (16) are orthogonal. Partitioning Equation (16), similarly to D_A and D_D yields

$$\begin{pmatrix} \Sigma_A & 0 \\ 0 & 0 \end{pmatrix} \tilde{X} + \tilde{Y} \begin{pmatrix} \Sigma_D & 0 \\ 0 & 0 \end{pmatrix} = \tilde{E}$$

$$\Leftrightarrow \quad \begin{pmatrix} \Sigma_A \tilde{X}_{11} + \tilde{Y}_{11} \Sigma_D & \Sigma_A \tilde{X}_{12} \\ \tilde{Y}_{21} \Sigma_D & 0 \end{pmatrix} = \begin{pmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ \tilde{E}_{21} & \tilde{E}_{22} \end{pmatrix}. \tag{17}$$

From Equation (17) one has the following theorem:

THEOREM 2. *Equation* (3) *is consistent if and only if*

$$\tilde{E}_{22} = U_{A2}^T E V_{D2} = 0, \tag{18}$$

with $U_A = (U_{A1}, U_{A2})$ *and* $V_D = (V_{D1}, V_{D2})$. *For a consistent equation* (3), *the submatrices* $\tilde{X}_{21}, \tilde{X}_{22}, \tilde{Y}_{12}, \tilde{Y}_{22}$ *of* $\tilde{X}$ *and* $\tilde{Y}$ *are arbitrary, and additional degrees of freedom can be found in*

$$\tilde{X}_{11} = (\tilde{x}_{ij}) \quad \text{and} \quad \tilde{Y}_{11} = (\tilde{y}_{ij}).$$

Elementwise, one has

$$\begin{pmatrix} \tilde{x}_{ij} \\ \tilde{y}_{ij} \end{pmatrix} = M_{ij}^+ \tilde{e}_{ij} + (I - M_{ij}^+ M_{ij}) Z_{ij} \tag{19a}$$

where $\Sigma_A = \operatorname{diag}(\alpha_i)$, $\Sigma_D = \operatorname{diag}(\delta_i)$, $\tilde{E} = (\tilde{e}_{ij})$, $M_{ij} = (\alpha_i, \delta_i)$, *and* Z_{ij} *are arbitrary.*

We may consider $\tilde{Y}_{11}$ *to be arbitrary, and* $\tilde{X}_{11}$ *then has to be chosen to be*

$$\tilde{X}_{11} = \Sigma_A^{-1} (\tilde{E}_{11} - \tilde{Y}_{11} \Sigma_D). \tag{19b}$$

If the condition (18) *is satisfied, the solution of Equation* (3) *may be written as*

$$X = V_A \begin{pmatrix} \Sigma_A^{-1} (\tilde{E}_{11} - Z_1 \Sigma_D) & \Sigma_A^{-1} \tilde{E}_{11} \\ Z_2 & Z_3 \end{pmatrix}$$

$$= V_A \begin{pmatrix} \Sigma_A^{-1} \\ 0 \end{pmatrix} U_{A1}^T (E - U_{A1} Z_1 \Sigma_D V_{D1}^T) V_D + V_{A2} (Z_2, Z_3) \tag{20a}$$

and

$$Y = \begin{pmatrix} Z_1 & Z_4 \\ \tilde{E}_{21}\Sigma_D^{-1} & Z_5 \end{pmatrix} U_D^T = \begin{pmatrix} Z_1 & Z_4 \\ U_{A2}^T E V_{D1}\Sigma_D^{-1} & Z_5 \end{pmatrix} U_D^T, \tag{20b}$$

where the matrices Z_1 *to* Z_5 *are arbitrary.*

The condition (18) and Equation (20) are equivalent to those given in Equations (2) to (4) in [1], without any explicit use of GI.

Again, Equation (17) leads naturally to a numerical algorithm for the solution of Equation (3)—if it is consistent. For inconsistent systems, a least squares type solution can also be found from Equations (17) and (19a), as all the transformations involved are orthogonal. The size of the residual will be the same as that of $\tilde{E}_{22}$.

The algorithm will be numerically unstable when any of the SV α_i or δ_i is small. Again, the small SV can be reset to zero and the resulting errors treated as residuals of Equation (3).

The SVD used to transform Equation (3) to (17) can be replaced by the less expensive QR decompositions [4] if the rank determinations of the matrices A^T and D are straightforward, and Equation (3) is then equivalent to

$$\begin{aligned} & R_A^T Q_A^T X + Y Q_D R_D = E \\ \Leftrightarrow \quad & R_A^T \tilde{X} + \tilde{Y} R_D = E, \end{aligned} \tag{21}$$

where $\tilde{X} = Q_A^T X$, $\tilde{Y} = Y Q_D$, with the matrices Q_A and Q_D orthogonal, and R_A and R_D upper triangular or trapezoidal.

An equivalent theory of consistency can be derived from Equation (21) instead of (17). A numerical algorithm for the solution of Equation (3) can then be derived, by considering the individual component of (21) in a rowwise or columnwise fashion.

Note also that special cases of Equation (1), e.g.

$$AXB + Y = E, \tag{22}$$

can be treated using SVD in a similar fashion as in this section.

5. $AXB = E$

It is obvious that consistency for Equations (2a) and (2b), [both in the form of Equation (4)] is necessary for the consistency of Equation (2).

To study the consistency of Equation (4), we decompose the matrices A and B by SVD:

$$U_A D_A V_A^T \cdot X \cdot U_B D_B V_B^T = E$$

$$\Leftrightarrow \quad \begin{pmatrix} \Sigma_A & 0 \\ 0 & 0 \end{pmatrix} \tilde{X} \begin{pmatrix} \Sigma_B & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ \tilde{E}_{21} & \tilde{E}_{22} \end{pmatrix} = \tilde{E}, \tag{23}$$

where $\tilde{X} = V_A^T X U_B$ and $\tilde{E} = U_A^T E V_B$. Equation (23) gives rise to the following theorem:

THEOREM 3. *Equation* (4) *is consistent if and only if*

$$(\tilde{E}_{21}, \tilde{E}_{22}) V_B^T = U_{A2}^T E = 0 \tag{24a}$$

and

$$U_A \begin{pmatrix} \tilde{E}_{12} \\ \tilde{E}_{22} \end{pmatrix} = E V_{B2} = 0, \tag{24b}$$

where $U_A = (U_{A1}, U_{A2})$ *and* $V_B = (V_{B1}, V_{B2})$. *The solution of Equation* (4), *if the condition* (24) *is satisfied, can be expressed as*

$$X = V_A \begin{pmatrix} \Sigma_A^{-1} \tilde{E}_{11} \Sigma_B^{-1} & Z_1 \\ Z_2 & Z_3 \end{pmatrix} U_B^T$$

$$= A^+ E B^+ + V_A \begin{pmatrix} 0 & Z_1 \\ Z_2 & Z_3 \end{pmatrix} U_B^T, \tag{25}$$

with the matrices Z_1, Z_2, *and* Z_3 *arbitrary.*

The condition (24) is equivalent to those derived in [2] and [5] for the consistency of Equation (4).

It is easy to see that the least squares solution of Equation (4) is possible from Equation (23) or (25), as the transformations in Equation (23) are all orthogonal. The choice of $Z_i = 0$ in Equation (25) will provide a minimum norm solution, X, and the residual of an inconsistent equation will be

$$\begin{pmatrix} 0 & \tilde{E}_{12} \\ \tilde{E}_{21} & \tilde{E}_{22} \end{pmatrix}.$$

The solution process in Equation (23) or (25) will be numerically unstable if any SV of the matrix A or B is small. Again, it may be necessary to reset small SV to zero and transfer the resulting errors to the RHS of Equation (3) to be treated as residuals.

Similarly to equation (21) the QR decomposition (or even the Guassian elimination process with suitable pivoting) can be applied to solve Equation (3), instead of the more expensive SVD in Equation (23), if the rank determinations of the matrices A and B are trouble-free.

6. $(AXB, FXG) = (E, H)$

Observe in Equation (2) that the matrices A and F (B^T and G^t) have the same number of columns, and apply the GSVD to the respective matrix pairs. Equation (2) is then equivalent to

$$U_1\Sigma_A X_1 \cdot X \cdot X_2^T \Sigma_B^T U_2^T = E, \tag{26a}$$

$$V_1\Sigma_F X_1 \cdot X \cdot X_2^T \Sigma_G^T V_2^T = H, \tag{26b}$$

where the matrices U_i and V_i are orthogonal, and X_i are nonsingular, as in Equations (7) to (9).

Define $\tilde{X} = X_1 \cdot X \cdot X_2^T$, $\tilde{E} = U_1^T E U_2$, and $\tilde{H} = V_1^T H V_2$; then Equation (27) is equivalent to

$$\Sigma_A \tilde{X} \Sigma_B^T = \tilde{E},$$

$$\Sigma_F \tilde{X} \Sigma_G^T = \tilde{H},$$

which in turn is equivalent to

$$\left(\begin{array}{ccc|c} I_A & & & \\ & S_A & & 0 \\ & & 0_A & \end{array}\right) \tilde{X} \left(\begin{array}{ccc} I_B & & \\ & S_B & \\ & & 0_B^T \\ \hline & 0 & \end{array}\right) = \tilde{E}, \tag{27a}$$

and

$$\left(\begin{array}{ccc|c} 0_F & & & \\ & S_F & & 0 \\ & & I_F & \end{array}\right)\tilde{X}\left(\begin{array}{ccc} 0_G^T & & \\ & S_G & \\ & & I_G \\ \hline & 0 & \end{array}\right) = \tilde{H}. \tag{27b}$$

Partitioning the matrices $\tilde{X}$, $\tilde{E}$, and $\tilde{H}$ in accordance with the Σ's, Equation (27) leads to:

THEOREM 4. *Equation* (2) *is consistent if and only if*:

$$\tilde{E}_{31}, \tilde{E}_{32}, \tilde{E}_{33}, \tilde{E}_{23}, \tilde{E}_{13} = 0, \tag{28a}$$

$$\tilde{H}_{13}, \tilde{H}_{12}, \tilde{H}_{11}, \tilde{H}_{21}, \tilde{H}_{31} = 0, \tag{28b}$$

$$\Delta = S_A^{-1}\tilde{E}_{22}S_B^{-1} = S_F^{-1}\tilde{H}_{22}S_G^{-1}. \tag{29}$$

The solution of Equation (2) *can be expressed as*

$$X = X_1^{-1}\tilde{X}X_2^{-T}, \tag{30}$$

where

$$\tilde{X} = \begin{pmatrix} \tilde{E}_{11} & \tilde{E}_{12}S_B^{-1} & Z_1 & Z_2 \\ S_A^{-1}\tilde{E}_{21} & \Delta & S_F^{-1}\tilde{H}_{23} & Z_3 \\ Z_4 & \tilde{H}_{32}S_G^{-1} & \tilde{H}_{33} & Z_5 \\ Z_6 & Z_7 & Z_8 & Z_9 \end{pmatrix}, \tag{31}$$

with the matrices Z_1 *to* Z_9 *arbitrary.*

Proof. Equations (28a) and (28b) are the consistency conditions of the individual equations in (27a) and (27b) respectively. [They are similar to the condition (24) in Theorem 3.]

Equation (29) is the result of matching the solutions of the equations involving $\tilde{X}_{22}$ in Equation (27).

Equation (31) is a trivial consequence of (27). ■

The algorithm for the solution of Equation (2) [suggested by Equation (27) or (31)] will be numerically unstable if any of the GSV is small, or if the matrix X_1 or X_2 is ill conditioned. [Recall the discussion on conditions of these matrices after Equation (11).]

Comments at the end of Section 3 concerning the possible instability related to the inversions of X_1 and X_2 also apply here.

From equation (26), the solution in (31) minimizes the 2- or F-norm of the residual of an inconsistent equation (2), but will not provide a minimum norm solution X, as X_1 and X_2 are not orthogonal.

Note that special cases of Equation (2), e.g.

$$AX = E,$$

$$XG = H,$$

can be similarly treated using the SVD.

7. DUALITY

It is not widely appreciated, but can be easily proved, that the adjoint equation of (1) is in the form of (2), and vice versa.

Starting from Equation (1) and using the Kronecker product, one has

$$AXB + CYD = E$$

$$\Leftrightarrow \quad (A\otimes B^T, C\otimes D^T)\cdot v\begin{pmatrix} X \\ Y \end{pmatrix} = v(E), \tag{32}$$

where the column vector $v(M)$ is formed by lining up successive rows of the matrix M and transposing. Considering the adjoint equation of Equation (32), one has

$$v(Z_1^T)^T \cdot (A\otimes B^T, C\otimes D^T) = v(Z_2^T)^T$$

iff

$$\begin{pmatrix} A^T\otimes B \\ C^T\otimes D \end{pmatrix} \cdot v(Z_1^T) = v(Z_2^T)$$

iff

$$\begin{cases} A^T Z_1^T B^T = Z_{21}^T, \\ C^T Z_1^T D^T = Z_{22}^T, \end{cases}$$

where

$$Z_2^T = \begin{pmatrix} Z_{21}^T \\ Z_{22}^T \end{pmatrix},$$

iff

$$BZ_1A = Z_{21},$$

$$DZ_1C = Z_{22}. \tag{33}$$

From Equations (32) and (33), one can derive the following conditions for consistency of Equations (1) and (2), using the duality property:

THEOREM 5. *Equation* (1) *is consistent if and only if*

$$BZA = 0 \text{ and } DZC = 0 \quad \Rightarrow \quad \operatorname{trace}(EZ) = 0.$$

Equation (2) *is consistent if and only if*

$$BZ_1A + GZ_2F = 0 \quad \Rightarrow \quad \operatorname{trace}(EZ_1 + HZ_2) = 0.$$

Proof. From Equations (32) and (33), (1) is consistent if

$$BZA = 0, \quad DZC = 0 \quad \Rightarrow \quad v(Z^T)^T v(E) = 0$$

and

$$v(Z^T)^T v(E) = \operatorname{trace}(EZ).$$

Similarly, starting from Equation (2), the second part of the theorem may be proved. ■

Obviously, Theorem 5 holds for special cases of Equations (1) and (2) [e.g. (3) and (4)] in their respective simplified forms.

8. CONCLUSIONS

In this paper, we have applied the GSVD to investigate the solution of the linear matrix equations (1) and (2). The special cases in (3) and (4) are treated using the SVD. Consistency conditions are derived and solutions for consistent equations are characterized. Possibilities of solving the equations in the least squares sense have also been discussed when appropriate. Additional conditions for the consistency of the equations are then derived, using the duality of equations of the form (1) and (2).

Although the paper is essentially a theoretical one, numerical algorithms for the solution of the equations are suggested and numerical considerations have always been kept in mind. More work—expecially numerical experimentation—needs to be done.

Finally, a potentially very powerful tool, the GSVD, has been around for nearly ten years, but applications have been surprisingly limited, compared to those of the SVD. This paper represents a modest attempt to redress the situation.

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